

ABSENCE OF PERCOLATION IN THE BERNOULLI BOOLEAN MODEL

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ABSTRACT. We consider the Bernoulli Boolean discrete percolation model on the d -dimensional integer lattice. We study sufficient conditions on the distribution of the radii of balls placed at the points of a Bernoulli point process for the absence of percolation, provided that the intensity of the underlying point process is small enough. We also study a Harris graphical procedure to construct, forward in time, particle systems with interactions of infinite range under the assumption that the corresponding generator admits a Kalikow-type decomposition. We do so by using the subcriticality of the boolean model of discrete percolation.

INTRODUCTION.

We study the Bernoulli Boolean discrete percolation model on the d -dimensional lattice \mathbb{Z}^d . This is a discrete percolation model which can be informally described as follows. Consider a Bernoulli point process \mathcal{X} with retention parameters $0 < p_x < 1$, $x \in \mathbb{Z}^d$, on the d -dimensional lattice \mathbb{Z}^d . This means that each site $x \in \mathbb{Z}^d$ is *present* or *absent* in \mathcal{X} with probability p_x or $1 - p_x$, respectively and independently of anything else. Each point of \mathcal{X} is the center of a ball of random radius in the metric induced by the L_1 norm. The random radii R_x , $x \in \mathbb{Z}^d$, are independent and independent of \mathcal{X} . We consider the *occupied region* which is defined as the subset of \mathbb{Z}^d obtained by taking the union of all random balls centred at the points of \mathcal{X} .

This model is the discrete counterpart of the Poisson Boolean model of continuum percolation. In the Poisson Boolean model a ball of random radius is centred at each point of a homogeneous Poisson point process with density λ on \mathbb{R}^d . The corresponding radii form an independent and identically distributed collection of non-negative random variables which are also independent of the point process. Denote by \mathcal{B} the union of these balls and by \mathcal{C} the connected component of \mathcal{B} containing the origin. Let R be one of the random radii and denote by \mathbf{P} the law governing the continuous boolean model. Also, denote by \mathbf{E} the corresponding expectation operator. In [7], Hall proved that for values of λ small enough, \mathcal{C} is almost surely bounded provided that $\mathbf{E}[R^{2d-1}]$ is finite. In [12], Meester and Roy proved that if $d \geq 2$ and $\mathbf{E}[R^d]$ is finite, then the expected number of balls in the occupied component which contains the origin is finite whenever λ is small enough if, and only if, $\mathbf{E}[R^{2d}]$ is finite. Also, they proved that if $\mathbf{E}[R^{2d-1}]$ is finite

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then $\mathbf{P}(\text{number of balls in any occupied component is finite}) = 1$ provided that λ is small enough. In [5], Gouere showed that the set \mathcal{C} is almost surely bounded for small enough λ if and only if $\mathbf{E}[R^d]$ is finite.

In this paper we prove that if $p_x = p \in (0, 1)$ for all x and the random radii $(R_x, x \in \mathbb{Z}^d)$ are i.i.d. random variables with finite d -moment, then the connected components arising in the discrete Boolean model are almost surely finite for sufficiently small values of p . We also prove that such behavior does not occur if the random radii have infinite d -moment. Then, using a coupling argument, we extend the result about subcriticality to the case where the values of p_x are not constant and the random radii are independent but not necessarily identically distributed. Then we use the result above about subcriticality to provide a graphical construction method for interacting particle systems with interactions of infinite range. In order to prove this result we assume that the generator of the particle system admits a Kalikow-type decomposition. Recently, this type of decomposition has been explored by Galves et al. in the context of perfect simulation of interacting particle systems with interactions of infinite range. More precisely, in [4] the authors exhibit a sufficient condition under which a Kalikow-type decomposition holds for the transition rates of interacting particle systems with interactions of infinite range. Namely, if the transition rates satisfy a continuity condition then the referred decomposition holds. For further details on Kalikow-type decompositions see [9] and [1].

This paper is organized as follows: In section 1 we describe the discrete boolean percolation model and state the main result of this work which says about the absence of percolation on the model described above. This result is proved in section 2 following ideas for the continuous boolean percolation model studied in [5]. In section 3 we extend the result in [8] on the graphical construction of interacting particle system with finite-range interaction to the case of interactions of infinite range, using the results in section 1 under mild assumption on the decay of the range of interaction.

1. DEFINITIONS, NOTATION AND MAIN RESULTS

Throughout this paper \mathbb{N}_0 will denote the set of non-negative integer numbers. We write $\|\cdot\|$ for the L_1 norm on \mathbb{Z}^d and $|A|$ for the cardinal number of any set $A \subset \mathbb{Z}^d$. Also, $B(x, r) = \{y \in \mathbb{Z}^d : \|y - x\| \leq r\}$ denotes the (close) ball of radius r centred at x and $S_r = \{x \in \mathbb{Z}^d : \|x\| = r\}$ denotes the sphere of radius r . For any set $A \subset \mathbb{Z}^d$, A^c stands for the complement of A .

If F denotes a cumulative distribution function, let F^{-1} be the generalized inverse of F defined by $F^{-1}(u) = \inf\{r \in \mathbb{R} : F(r) \geq u\}$ where $u \in [0, 1]$. If X and Y are two stochastic elements equally distributed, we write $X \stackrel{D}{=} Y$.

A *Bernoulli point process* on \mathbb{Z}^d with retention parameters $\mathbf{p} = (p_x : x \in \mathbb{Z}^d)$, where $0 < p_x < 1$ for all $x \in \mathbb{Z}^d$, is a family of independent $\{0, 1\}$ -valued random variables $\mathcal{X} = (X_x : x \in \mathbb{Z}^d)$ such that p_x is the probability of the event $\{X_x = 1\}$. Identify the family of random variables \mathcal{X} with the random subset \mathcal{P} of \mathbb{Z}^d defined by $\mathcal{P} = \{x \in \mathbb{Z}^d : X_x = 1\}$ whose distribution is a product measure whose marginals at each site x are Bernoulli distribution of parameter p_x .

By a *Bernoulli marked point process* on \mathbb{Z}^d we mean a pair $(\mathcal{X}, \mathcal{R})$ formed by a Bernoulli point process \mathcal{X} on \mathbb{Z}^d and a family of independent \mathbb{N}_0 -valued random variables $\mathcal{R} = (R_x : x \in \mathbb{Z}^d)$ called marks. We assume that these marks are independent of the point process \mathcal{X} .

Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process on \mathbb{Z}^d . Let p_x be the retention parameter of the random variable X_x and let ν_x be the probability function of the random variable R_x . If there exists a value $p \in (0, 1)$ and a probability function ν on \mathbb{N}_0 such that $p_x = p$ and $\nu_x = \nu$ for every $x \in \mathbb{Z}^d$ we say that the marked point process $(\mathcal{X}, \mathcal{R})$ is *spatially homogeneous with retention parameter p and marks distributed according to ν* .

Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process on \mathbb{Z}^d with retention parameters $\mathbf{p} = (p_x : x \in \mathbb{Z}^d)$ and marks distributed according to a family of probability functions $\mathbf{n} = (\nu_x : x \in \mathbb{Z}^d)$. We denote by $\mathbf{P}_{\mathbf{p}, \mathbf{n}}$ and $\mathbf{E}_{\mathbf{p}, \mathbf{n}}$ respectively the probability measure and the expectation operator induced by $(\mathcal{X}, \mathcal{R})$. If $(\mathcal{X}, \mathcal{R})$ is spatially homogeneous with retention parameter p and marks distributed according to the probability function ν , we denote by $\mathbf{P}_{p, \nu}$ and $\mathbf{E}_{p, \nu}$ respectively the probability measure and the expectation induced by $(\mathcal{X}, \mathcal{R})$.

Let $(\mathcal{X}, \mathcal{R})$ and $(\mathcal{X}', \mathcal{R}')$ be two marked point process on \mathbb{Z}^d defined on the same probability space. If

$$X_x \leq X'_x \quad \text{and} \quad R_x \leq R'_x, \quad x \in \mathbb{Z}^d,$$

we say that $(\mathcal{X}, \mathcal{R})$ is *dominated* by $(\mathcal{X}', \mathcal{R}')$ and we denote this by $(\mathcal{X}, \mathcal{R}) \preceq (\mathcal{X}', \mathcal{R}')$.

Random Graphs and Percolation. Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process on \mathbb{Z}^d . Then we define an associated random graph $\mathcal{G}(\mathcal{X}, \mathcal{R}) = (\mathbb{Z}^d, \mathcal{E})$ as the undirected random graph with vertex set \mathbb{Z}^d and edge set \mathcal{E} defined by the condition $\{x, y\} \in \mathcal{E}$ if, and only if, $X_x = 1$ and $y \in B(x, R_x)$ or $X_y = 1$ and $x \in B(y, R_y)$.

A *path* on $\mathcal{G}(\mathcal{X}, \mathcal{R})$ is a sequence of distinct vertex x_0, x_1, \dots, x_n with $x_{i-1} \neq x_i$ such that $\{x_{i-1}, x_i\} \in \mathcal{E}$, $i = 1, \dots, n$.

A set of vertex $C \subset \mathbb{Z}^d$ is connected if, for all pair of distinct vertex x and y in C , there exists a path on $\mathcal{G}(\mathcal{X}, \mathcal{R})$ using vertices only from C , starting at x and ending at y . The connected components of the graph $\mathcal{G}(\mathcal{X}, \mathcal{R}) = (\mathbb{Z}^d, \mathcal{E})$ are its maximal connected subgraphs.

The cluster $C(x)$ of vertex x is the connected component of the graph $\mathcal{G}(\mathcal{X}, \mathcal{R})$ containing x . Define the Percolation as follows:

$$(1) \quad [\text{Percolation}] := \bigcup_{x \in \mathbb{Z}^d} \{|C(x)| = \infty\}.$$

Phase transition. Consider the Bernoulli Boolean discrete percolation model introduced above. Then replace the random radii in this model by the deterministic radius 0. What we get is the independent site percolation model. It is well known (see Grimmett [6], page 25) that the critical parameter for this last model is a positive number $p_c^{\text{site}}(\mathbb{Z}^d) < 1$. Then, a coupling argument shows that for any $p > p_c^{\text{site}}(\mathbb{Z}^d)$ there is percolation for the discrete Boolean model. Thus we focus our attention in the subcritical regime.

Now we state the main result of this work.

Theorem 1. *Let $(\mathcal{X}, \mathcal{R})$ be a spatially homogeneous marked point process on \mathbb{Z}^d with retention parameter p and marks distributed according to a probability function ν . If $\sum_{r \geq 1} r^d \nu(r) < \infty$, then there exists $p_0 > 0$ such that $\mathbf{P}_{p, \nu}(\text{Percolation}) = 0$ for all $p \leq p_0$.*

Indeed, a similar result holds if we only assume that the values of p_x are uniformly bounded and the family of random radii are independent, but not identically distributed.

Theorem 2. *Let $(\mathcal{X}, \mathcal{R})$ be a marked point process on \mathbb{Z}^d with retention parameters $\mathbf{p} = (p_x : x \in \mathbb{Z}^d)$, and random marks $R_x, x \in \mathbb{Z}^d$ distributed according to a family of probability functions $\mathbf{n} = (\nu_x : x \in \mathbb{Z}^d)$. If*

$$(2) \quad \lim_{r \rightarrow \infty} \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{\mathbf{p}, \mathbf{n}}(R_x \leq r) = 1$$

and

$$(3) \quad \sum_r r^d \left(\inf_{x \in \mathbb{Z}^d} \mathbf{P}_{\mathbf{p}, \mathbf{n}}(R_x \leq r) - \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{\mathbf{p}, \mathbf{n}}(R_x \leq r-1) \right) < \infty,$$

then there exists $p_0 > 0$ such that $\mathbf{P}_{\mathbf{p}, \mathbf{n}}(\text{Percolation}) = 0$ for any family of retention parameters $\mathbf{p} = (p_x : x \in \mathbb{Z}^d)$ such that $\sup_{x \in \mathbb{Z}^d} p_x \leq p_0$.

Remark 3. *We claim that hypothesis (2) in Theorem 2 above is equivalent to assume the existence of a random variable R such that each random variable in \mathcal{R} is stochastically dominated by R . Indeed, let U be a uniform random variable on $[0, 1]$. Then, define R as follows:*

$$(4) \quad R = \sum_{r \geq 1} r \cdot \mathbf{1} \left\{ \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{\mathbf{p}, \mathbf{n}}(R_x \leq r-1) < U \leq \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{\mathbf{p}, \mathbf{n}}(R_x \leq r) \right\}.$$

We readily verify that R is a random variable and that each random variable in \mathcal{R} is stochastically dominated by R . By hypothesis (3) we have $\mathbf{E}[R^d] < \infty$. Now, using this stochastic domination we may construct, in a common probability space, two marked point processes. For that purpose, let $(U_x : x \in \mathbb{Z}^d)$ be a family of i.i.d. random variables, each one uniformly distributed on $[0, 1]$. Then, set $\hat{R}_x = F_{R_x}^{-1}(U_x)$ and $\hat{R}^x = F_R^{-1}(U_x)$, where F_{R_x} and F_R are the cumulative distribution function of R_x and R respectively. It follows that $\hat{R}_x \stackrel{D}{=} R_x$, $\hat{R}^x \stackrel{D}{=} R$ and $\hat{R}_x \leq \hat{R}^x$. Since $\hat{R}_x \leq \hat{R}^x$, we get that $B(x, \hat{R}_x) \subset B(x, \hat{R}^x)$. Then, $(\mathcal{X}, \mathcal{R}_1) \preceq (\mathcal{X}, \mathcal{R}_2)$, where $\mathcal{R}_1 = (\hat{R}_x : x \in \mathbb{Z}^d)$ and $\mathcal{R}_2 = (\hat{R}^x : x \in \mathbb{Z}^d)$. Since $(\mathcal{X}, \mathcal{R}) \stackrel{D}{=} (\mathcal{X}, \mathcal{R}_1)$ and $\mathbf{E}[R^d] < \infty$, Theorem 2 is a simple consequence of Theorem 1.

Example 4. *Observe that condition (3) in Theorem 2 turns out to be slightly stronger than requiring $\sup_{x \in \mathbb{Z}^d} \mathbf{E}_{\mathbf{p}, \mathbf{n}}[R_x^d] < \infty$. Note that if the random radii $R_x, x \in \mathbb{Z}^d$ are i.i.d random variables, then condition (3) in Theorem 2 becomes $\mathbf{E}_{\mathbf{p}, \mathbf{n}}[R^d] < \infty$, where R is a random variable distributed as R_x for some $x \in \mathbb{Z}^d$. The example below shows that it is possible to construct a sequence of random variables $(R_n)_{n \in \mathbb{N}}$ in a common probability space such that*

$$(5) \quad \sup_{n \in \mathbb{N}} \mathbf{E}[R_n] < \infty \text{ with } \mathbf{E}[R] = \infty,$$

where R is a random variable such that $\mathbf{P}(R \leq r) = \inf_{n \in \mathbb{N}} \mathbf{P}(R_n \leq r)$ and \mathbf{E} is the corresponding expectation operator.

Let $(R_n)_{n \geq 2}$ be a sequence of random variables with distribution function

$$\begin{aligned} F_n(x) &= \left(1 - \frac{3}{4n}\right) \mathbf{1}\{0 \leq x < 1\} \\ &+ \left(\frac{1}{4n(n-1)}(x-1) + 1 - \frac{3}{4n}\right) \mathbf{1}\{1 \leq x < n\} \\ &+ \left(1 - \frac{1}{2n}\right) \mathbf{1}\{n \leq x < n+1\} \\ &+ \mathbf{1}\{n+1 \leq x\}. \end{aligned}$$

We readily check that

$$\sup_{n \geq 2} \mathbf{E}[R_n] < \infty.$$

The distribution function $F(x) = \inf_{n \in \mathbb{N}} F_n(x)$ is given by

$$F(x) = \sum_{n \geq 2} \left(1 - \frac{1}{2n}\right) \mathbf{1}\{n \leq x < n+1\}.$$

Finally, note that if R is a random variable with distribution function F as above, then

$$\mathbf{E}[R] \leq \sum_{n \geq 2} \frac{1}{2n} = +\infty.$$

This example shows that, with our techniques, the hypothesis in Theorem 2 can not be weakened.

Complete Coverage. We complement the result of Theorem 1 by establishing a sufficient condition for complete coverage of the space \mathbb{Z}^d . For any $A \subset \mathbb{Z}^d$, define $\Lambda(A) = \bigcup_{x \in A \cap \mathcal{P}} B(x, R_x)$.

Theorem 5. *Let $(\mathcal{X}, \mathcal{R})$ be a spatially homogeneous marked point process on \mathbb{Z}^d with retention parameter p and marks distributed according to the probability function ν . If $\sum_{r \geq 1} r^d \nu(r) = \infty$, then for any $p \in (0, 1]$, $\Lambda(\mathbb{Z}^d) = \mathbb{Z}^d$.*

1.1. Particle systems with interactions of infinite range. Let S be a finite (or countable) set and let $S^{\mathbb{Z}^d}$ be the set of mappings $\sigma : \mathbb{Z}^d \rightarrow S$. Give S the discrete topology and $S^{\mathbb{Z}^d}$ the product topology. The measurable sets of $S^{\mathbb{Z}^d}$ are the Borel sets. The elements of S are called *spins* or *particles*. $S^{\mathbb{Z}^d}$ is called the configuration space and its elements are in general written as σ, η, ξ, \dots . For each $x \in \mathbb{Z}^d$, $\sigma(x)$ denotes the spin value of configuration σ at site x . For each $A \subset \mathbb{Z}^d$, $\sigma(A) \in S^A$ denotes the restriction of configuration σ to A .

A particle system with interactions of infinite range is a Markov process on $S^{\mathbb{Z}^d}$ whose generator is defined on cylinder functions by

$$(6) \quad Lf(\sigma) = \sum_{x \in \mathbb{Z}^d} \sum_{s \in S} c_x(s, \sigma) [f(\sigma_{x,s}) - f(\sigma)],$$

where $\sigma_{x,s} \in S^{\mathbb{Z}^d}$ is defined by $\sigma_{x,s}(x) = s$, $\sigma_{x,s}(y) = \sigma(y)$ if $y \neq x$. Here $c_x(s, \sigma) > 0$ is the intensity for a jump $\sigma \rightarrow \sigma_{x,s}$ and it depends on x and the whole spin configuration σ . Kalikow-type decomposition. We assume that the following *Kalikow-type decomposition* for the jump intensities holds:

$$(7) \quad c_x(s, \sigma) = M_x p_x(s|\sigma),$$

where $M_x > 0$ and

$$(8) \quad p_x(s|\sigma) = \sum_{r \geq 0} \nu_x(r) p_x^{[r]}(s|\sigma).$$

Here, $\nu_x(\cdot)$ is a probability function on \mathbb{N}_0 and $p_x^{[r]}(\cdot|\sigma)$ is a probability function on S which depends on σ only through $\{\sigma(y) : y \in B(x, r)\}$. For further details on this kind of decomposition see [9] and [1].

Here and for the rest of the paper we will assume that

$$(9) \quad 0 < M_* := \inf_{x \in \mathbb{Z}^d} M_x \leq M^* := \sup_{x \in \mathbb{Z}^d} M_x < \infty$$

and

$$(10) \quad \sum_{r \geq 1} r^d \sup_{x \in \mathbb{Z}^d} \nu_x(r) < \infty.$$

Now we can state the result about existence of interacting particle systems with interactions of infinite range.

Theorem 6. *Let $\{c_x(s, \sigma) : x \in \mathbb{Z}^d, s \in S, \sigma \in S^{\mathbb{Z}^d}\}$ be a family of jump intensities satisfying the Kalikow-type decomposition described in (7) and (8). Let assumptions (9) and (10) hold. Then, for each initial spin configuration η , there exists an almost surely unique interacting particle system $(\sigma_t^\eta)_{t \geq 0}$ with generator*

$$(11) \quad Lf(\sigma) = \sum_{x \in \mathbb{Z}^d} \sum_{s \in S} \sum_{r \geq 0} M_x \nu_x(r) p_x^{[r]}(s|\sigma) [f(\sigma_{x,s}) - f(\sigma)].$$

In the rest of this section we present the sketch of the proof of Theorem 6 based on an extension, to the case of interactions with infinite range, of ideas developed by Harris for the case of finite range interactions. See [8] for further details.

Harris graphical construction. The probability space where the Markov processes $(\sigma_t^\eta : t \geq 0)$ will be constructed is the space generated by a family $(\mathcal{T}, \mathcal{K}, \mathcal{U}) = \{(\mathcal{T}_x, \mathcal{K}_x, \mathcal{U}_x) : x \in \mathbb{Z}^d\}$ of mutually independent marked Poisson point processes on the time line $[0, \infty)$. For each $x \in \mathbb{Z}^d$, the Poisson process $\mathcal{T}_x = (T_{x,n} : n \in \mathbb{N})$ is homogeneous with rate M_x , $\mathcal{K}_x = (K_{x,n} : n \in \mathbb{N})$ is a sequence of i.i.d. random variables with common law ν_x on \mathbb{N}_0 and $\mathcal{U}_x = (U_{x,n} : n \in \mathbb{N})$ is a sequence of i.i.d. uniform random variables on $[0, 1]$. Moreover, for each $x \in \mathbb{Z}^d$, $\mathcal{T}_x, \mathcal{K}_x$ and \mathcal{U}_x are mutually independent.

For each $\eta \in S^{\mathbb{Z}^d}$, we construct a process $(\sigma_t^\eta : t \geq 0)$ with generator (11) and initial condition η at time 0 as a function of the family $(\mathcal{T}, \mathcal{K}, \mathcal{U})$. Roughly speaking, the process $(\sigma_t^\eta : t \geq 0)$ is constructed as follows. Initially, $\sigma_0^\eta := \eta$. Then, at the time epoch $t \in \mathcal{T}_x$, the spin value at site x is updated in the following way: if $t = T_{x,n}$, then sample the range of interaction using the random variable $K_{x,n} \in \mathcal{K}_x$. If $K_{x,n} = r$, then the spin value at site x is updated by a random variable $W_x(\sigma_{t-}^\eta)$ with law $p_x^{[r]}(\cdot | \sigma_{t-}^\eta)$:

$$(12) \quad \sigma_t^\eta = \sigma_{x, W_x(\sigma_{t-}^\eta)}^\eta.$$

The random variable $W_x(\sigma_{t-}^\eta)$ is constructed as a function of the uniform random variable $U_{x,n} \in \mathcal{U}_x$.

Since there are infinitely many Poisson processes the main difficulty in the construction described above is that in general there will be infinitely many jumps in each interval of time.

The key of the Harris graphical construction [8] is to show that during a certain interval of time $[0, t_0]$, \mathbb{Z}^d can be partitioned into a countable number of finite random sets, called islands, with no interaction between islands. For this purpose we introduce a family of random graphs containing all the information concerning the interactions needed in each interval of time $[\tau, t]$.

Harris random graph. Fix $t > 0$. For each $0 \leq \tau \leq t$, let $\mathcal{G}_{\tau,t} = (\mathbb{Z}^d, \mathcal{E}_{\tau,t})$ be the undirected random graph with vertex set \mathbb{Z}^d and edge set $\mathcal{E}_{\tau,t}$ defined by $\{x, y\} \in \mathcal{E}_{\tau,t}$ if, and only if, $(\mathcal{T}_x \cup \mathcal{T}_y) \cap (\tau, t] \neq \emptyset$ and if there exists $t' \in (\mathcal{T}_x \cup \mathcal{T}_y) \cap (\tau, t]$ such that: (i) $y \in B(x, K_{x,n})$ if $t' = T_{x,n}$ or (ii) $x \in B(y, K_{y,m})$ if $t' = T_{y,m}$.

Note that the presence of an edge $\{x, y\} \in \mathcal{E}_{\tau,t}$ indicates that a Poisson epoch has caused x to look at y in order to figure out how to update its spin value, or, has caused y to look at x in order to figure out how to update its spin value. Conversely, if there is no edge between x and y then none of them have looked at each other. The last observation implies that sites in different components of the resulting random graph do not influence each other during the time interval $(\tau, t]$. Hence their evolutions can be computed separately.

In Section 3 we give the proof of the following result.

Theorem 7. *Let assumptions (9) and (10) hold. Then, there exists $t_0 > 0$ such that for any $t \leq t_0$, the connected components of the Harris random graph $\mathcal{G}_{0,t}$ are, almost surely, finite.*

Using Theorem 7 we can show that during the time interval $[0, t_0]$, where $t_0 > 0$ is deterministic and small enough, \mathbb{Z}^d can be partitioned into a countable number of finite islands, with no interaction between them: $\mathbb{Z}^d = \cup_{\ell \in \mathbb{N}} C_\ell$, where for each $\ell \in \mathbb{N}$, C_ℓ is a

finite set which is itself a deterministic function of the family of Poisson processes on $[0, t_0]$ and the marks \mathcal{K} . The collection $(C_\ell, \ell \in \mathbb{N})$ has the additional property that $C_{\ell_1} \cap C_{\ell_2} = \emptyset$ for every $\ell_1 \neq \ell_2$. During the time interval $[0, t_0]$ the process is constructed separately in each region C_ℓ independently of everything else. The details of this construction are given in subsection 4.1.

2. PROOF OF THEOREM 1

The proof of Theorem 1 will be divided into two steps. In the first step we introduce two families of events, $G(x, r)$ and $H(r)$, in order to study the diameter of the cluster $C(0)$. The family of events $G(x, r)$ is helpful to understand the behavior of the cluster $C(0)$ on the subgraph of $\mathcal{G}(\mathcal{X}, \mathcal{R})$ induced by the point process on $B(0, 10r)$. The family of events $H(r)$ provides a way to take care of the influence of the point process $(\mathcal{X}, \mathcal{R})$ from the exterior of the ball $B(0, r)$. Our aim in this step is to show that the probability of the percolation event can be controlled by the probabilities of the events $G(0, r)$. In the second step we will show that if the radii are not too large, then the occurrence of the event $G(0, r_1)$ implies the occurrence of two independent events $G(x, r_2)$ and $G(x', r_2)$ where $r_1 = 10r_2$. Our aim in this step is to show that the probability of the events $G(0, r_1)$ can be bounded by the square of the probability of the events $G(0, r_2)$ plus a quantity that goes to zero when r_1 goes to infinity. This provides a way to take care of the probabilities of the events $G(0, r)$ that allows us to show that for p small enough, $\mathbf{P}_{p, \nu}(G(0, r))$ goes to zero when r goes to infinity.

2.1. Controlling the diameter of the cluster of the origin. For each $x \in \mathbb{Z}^d$, let $D_x = \inf\{r \geq 0 : C(x) \subset B(x, r)\}$. The percolation event is equivalent to the event $\bigcup_{x \in \mathbb{Z}^d} \{D_x = \infty\}$. By translation invariance of spatially homogeneous marked point processes, the probability of the events $\{D_x > r\}$ does not depend on x . Therefore, the proof of Theorem 1 is reduced to show the existence of $p_0 > 0$ such that $\lim_{r \rightarrow \infty} \mathbf{P}_{p, \nu}(D > r) = 0$ for all $p < p_0$, where D is the random variable D_x at the origin.

We define two families of events to study the diameter of the cluster $C(0)$. The family of events $G(x, r)$. Let B be a subset of \mathbb{Z}^d . Denote by $\mathcal{G}[B]$ the subgraph of $\mathcal{G}(\mathcal{X}, \mathcal{R})$ induced by B . Let A be a non-empty subset of \mathbb{Z}^d contained in B and let $x \in A$. We say that x is *disconnected from the exterior of A inside B* if the connected component of $\mathcal{G}[B]$ containing x is contained in A . Now we introduce the events $G(x, r)$. Let $x \in \mathbb{Z}^d$ and let $r \in \mathbb{N}$, we say that $G(x, r)$ does not occurs if x is disconnected from the exterior of $B(x, 8r)$ inside $B(x, 10r)$.

The family of events $H(r)$. For each $r \in \mathbb{N}$, define

$$(13) \quad H(r) = \left\{ \exists x \in \mathcal{P} \cap B(0, 10r)^c : R_x > \frac{\|x\|}{10} \right\}.$$

The relation between the diameter of the cluster at the origin and the families of events defined above is established in the following lemma.

Lemma 8. *The following assertion holds for all $r \in \mathbb{N}$:*

$$(14) \quad G(0, r)^c \cap H(r)^c \subset \{D \leq 8r\}.$$

Proof of Lemma 8. If the event $H(r)$ does not occur, then there are no sites of the point process with norm greater than $10r$ connected to $B(0, 9r)$. Indeed, assume that $H(r)$ does not occur. Then for every $x \in \mathcal{P} \cap B(0, 10r)^c$ we have $\|x\| - R_x \geq \frac{9}{10}\|x\| > 9r$. Using the triangle inequality it is easy to see that $\|y\| \geq \|x\| - R_x > 9r$ for all $y \in B(x, R_x)$. If $G(0, r)$ does not occur, then 0 is isolated from the exterior of $B(0, 8r)$. If, in addition, the event $H(r)$ does not occur, then the balls $B(x, R_x)$ with $x \in \mathcal{P} \cap B(0, 10r)^c$ do not help to connect the origin to the complement of $B(0, 8r)$. Thus $D \leq 8r$. \square

From (14) we get

$$(15) \quad \mathbf{P}_{p,\nu}(D > 8r) \leq \mathbf{P}_{p,\nu}(G(0, r)) + \mathbf{P}_{p,\nu}(H(r)).$$

Notice that $\lim_{r \rightarrow \infty} \mathbf{P}_{p,\nu}(H(r)) = 0$ for all $p \in (0, 1)$. This is obvious because $H(r+1) \subset H(r)$ for all $r \in \mathbb{N}$ and $\bigcap_{r \in \mathbb{N}} H(r) = \emptyset$.

2.2. Controlling the probabilities of the events $G(0, r)$. To take care of the probabilities $\mathbf{P}_{p,\nu}(G(0, r))$ we introduce another family of events.

The family of events $\tilde{H}(r)$. For each $r \in \mathbb{N}$, we define

$$(16) \quad \tilde{H}(r) = \{\exists x \in \mathcal{P} \cap B(0, 100r) : R_x \geq r\}.$$

Lemma 9. *The following inclusion holds for all $r \in \mathbb{N}$:*

$$(17) \quad G(0, 10rd) \cap \tilde{H}(rd)^c \subset \left(\bigcup_{x \in S_{10d}} G(rx, rd) \right) \cap \left(\bigcup_{x \in S_{80d}} G(rx, rd) \right).$$

Proof of Lemma 9. Fix $r \in \mathbb{N}$. First, assume that the event $G(0, 10rd)$ occurs but the event $\tilde{H}(rd)$ does not occur. Since $G(0, 10rd)$ occurs we can go from the origin to the complement of the ball $B(0, 80rd)$ just using balls $B(x, R_x)$ centred at points from $\mathcal{P} \cap B(0, 100rd)$. In this way, we can go from the sphere S_{10rd} to the sphere S_{80rd} . One of this balls, let say $B(x_*, R_{x_*})$, touches S_{10rd} . Since the sphere S_{10rd} is a subset of $\bigcup_{x \in S_{10d}} B(rx, rd)$ (see Proposition 18 in the Appendix), we get that this ball touches a ball of the form $B(rk, rd)$ for some k in S_{10d} .

Now we shall prove that, for this k , the event $G(rk, rd)$ occurs. It is easy to see that we can go from $B(rk, rd)$ to the complement of $B(rk, 8rd)$ just using balls of the form $B(x, R_x)$ centred at points from $\mathcal{P} \cap B(0, 100rd)$. Since $\tilde{H}(rd)$ does not occur, the radius of any such ball is less than rd . Then we can go from $B(rk, rd)$ to the complement of $B(rk, 8rd)$ just using balls of the form $B(x, R_x)$ centred at points from $\mathcal{P} \cap B(rk, 10rd)$. In other words, the event $G(rk, rd)$ occurs. Then, the event $\bigcup_{x \in S_{10d}} G(rx, rd)$ does occur. The proof that the event $\bigcup_{x \in S_{80d}} G(rx, rd)$ does occur follows in the same lines. \square

The event on the right side of (17) is the intersection of two events. The first depends on what happens inside $B(0, 20rd)$. The other event only depends on what happens in the

region $B(0, 70rd)^c$. Then, these two events are independent. By translation invariance of spatially homogeneous marked point processes we get

$$(18) \quad \mathbf{P}_{p,\nu}(G(0, 10rd)) \leq |S_{10d}| |S_{80d}| \mathbf{P}_{p,\nu}(G(0, rd))^2 + \mathbf{P}_{p,\nu}(\tilde{H}(rd)).$$

Lemma 10. *There exist positive constants C_2 and C_3 , which depends only on the dimension d , such that, for any $r \in \mathbb{N}$, the following inequalities hold:*

$$(19) \quad \mathbf{P}_{p,\nu}(G(0, r)) \leq p C_2 r^d,$$

$$(20) \quad \mathbf{P}_{p,\nu}(\tilde{H}(r)) \leq p C_3 \mathbf{E}_{p,\nu} [R^d \mathbf{1}\{R \geq r\}].$$

Proof of Lemma 10. It is a simple geometric fact that there exists a positive constant C which depends only on the dimension d such that $|B(0, r)| \leq C r^d$.

Let $r \in \mathbb{N}$. A simple computation shows that

$$(21) \quad \begin{aligned} \mathbf{P}_{p,\nu}(G(0, r)) &\leq \mathbf{P}_{p,\nu}(\exists x \in \mathcal{P} \cap B(0, 10r)) \\ &\leq p |B(0, 10r)|. \end{aligned}$$

The inequality (19) is satisfied with $C_2 = 10^d C$.

To show (20) we note that $\tilde{H}(r) = \mathbf{1}\{X \geq 1\}$, where X is a random variable defined by

$$X = \sum_{x \in B(0, 100r)} \mathbf{1}\{x \in \mathcal{P}\} \mathbf{1}\{R_x \geq r\}.$$

We have

$$\begin{aligned} \mathbf{P}_{p,\nu}(\tilde{H}(r)) &\leq \mathbf{E}_{p,\nu} [X] \\ &= \sum_{x \in B(0, 100r)} p \mathbf{P}_{p,\nu}(R_x \geq r) \\ &= p |B(0, 100r)| \mathbf{P}_{p,\nu}(R \geq r) \\ &\leq p C_3 \mathbf{E}_{p,\nu} [R^d \mathbf{1}\{R \geq r\}], \end{aligned}$$

where $C_3 = 100^d C$. The first equality follows from the independence between \mathcal{P} and \mathcal{R} and the second equality follows from the fact that the random variables $(R_x, x \in \mathbb{Z}^d)$ are identically distributed. \square

2.3. Proof of Theorem 1. By (15), the proof of Theorem 1 is reduced to show the existence of $p_0 > 0$ such that there exists an increasing sequence $(r_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{n \rightarrow \infty} \mathbf{P}_{p,\nu}(G(0, r_n)) = 0$ for any $p < p_0$. For this reason we need the following lemma.

Lemma 11. *Let f and g be two functions from \mathbb{N} to \mathbb{R}_+ satisfying the following conditions: (i) $f(r) \leq 1/2$ for all $r \in \{1, \dots, 10\}$; (ii) $g(r) \leq 1/4$ for all $r \in \mathbb{N}$; (iii) for all $r \in \mathbb{N}$:*

$$(22) \quad f(10r) \leq f^2(r) + g(r).$$

If $\lim_{r \rightarrow \infty} g(r) = 0$, then $\lim_{n \rightarrow \infty} f(10^n r) = 0$ for each $r \in \{1, \dots, 10\}$.

Proof of Lemma 11. For each $n \in \mathbb{N}$, let $F_n = \max_{1 \leq r \leq 10} f(10^n r)$ and let $G_n = \max_{1 \leq r \leq 10} g(10^n r)$. Using (22) and hypothesis (i) and (ii) we may conclude, by means of the induction principle that, for each $n \in \mathbb{N}$, $F_n \leq 1/2$ and

$$(23) \quad F_n \leq \frac{1}{2^{n+1}} + \sum_{j=0}^{n-1} \frac{1}{2^j} G_{n-1-j}.$$

Since $g(10^n r)$ goes to zero as $n \rightarrow \infty$ we have that $G_n \rightarrow 0$ when $n \rightarrow \infty$. By (23), we obtain that $F_n \rightarrow 0$ when $n \rightarrow \infty$. \square

Consider the functions $f(r) = C_1 \mathbf{P}_{p,\nu}(G(0, rd))$ and $g(r) = C_1 \mathbf{P}_{p,\nu}(\tilde{H}(rd))$, where $C_1 = |S_{10d}| |S_{80d}|$. By (18), it follows that

$$(24) \quad f(10r) \leq f^2(r) + g(r).$$

By condition $\mathbf{E}_{p,\nu}[R^d] = \sum_{r \geq 1} r^d \nu(r) < \infty$ and (20), we have that $\lim_{r \rightarrow \infty} g(r) = 0$ for any p .

We show that there exists $p_0 > 0$ such that if $p < p_0$ then $f(r) \leq 1/2$, $1 \leq r \leq 10$ and $g(r) \leq 1/4$, $r \in \mathbb{N}$.

Set

$$p_0 = \min((2C_1 C_2 (10d)^d)^{-1}, (4C_1 C_3 \mathbf{E}_{p,\nu}[R^d])^{-1}).$$

By condition $\mathbf{E}_{p,\nu}[R^d] < \infty$, we get $p_0 > 0$.

Let $p > 0$ be such that $p \leq p_0$. It follows from (19) that

$$f(r) \leq \frac{1}{2} \left(\frac{r}{10} \right)^d.$$

Thus we have that if $0 < p \leq p_0$, then $\max_{1 \leq r \leq 10} f(r) \leq 1/2$.

By (20), we get

$$g(r) \leq \frac{1}{4}.$$

Finally, by Lemma 11, we may conclude that $\lim_{n \rightarrow \infty} f(10^n r) = 0$ for each $r \in \{1, \dots, 10\}$. In particular,

$$\lim_{n \rightarrow \infty} f(10^n) = \lim_{n \rightarrow \infty} C_1 \mathbf{P}_{p,\nu}(G(0, 10^n d)) = 0.$$

\square

We finish this section by proving the complete coverage of \mathbb{Z}^d under the assumption $\mathbf{E}_{p,\nu}[R^d] = \infty$.

2.4. Proof of Theorem 5. We prove the equivalent statement that, for all $r \in \mathbb{N}$, the following assertion holds:

$$\mathbf{P}_{p,\nu}(\exists x \in \mathcal{P} : B(0, r) \subset B(x, R_x)) = 1.$$

If $R_x > \|x\| + r$, then $B(0, r) \subset B(x, R_x)$. Hence,

$$\mathbf{P}_{p,\nu}(\exists x \in \mathcal{P} : B(0, r) \subset B(x, R_x)) \geq \mathbf{P}_{p,\nu}(\exists x \in \mathcal{P} : R_x > \|x\| + r)$$

Let A_k be the event defined by $A_k = \{\exists x \in \mathcal{P} \cap S_k : R_x > k + r\}$. It is clear that the events A_k are independent and that $\mathbf{P}_{p,\nu}(A_k) = p|S_k|\mathbf{P}_{p,\nu}(R > k + r)$. Note that

$$\begin{aligned} \sum_{k \geq 0} \mathbf{P}_{p,\nu}(A_k) &= p \sum_{k \geq 0} |S_k| \mathbf{P}_{p,\nu}(R > k + r) \\ (25) \qquad \qquad &= \sum_{k \geq 0} |B_k| \mathbf{P}_{p,\nu}(R = k + r + 1). \end{aligned}$$

Since $\mathbf{E}_{p,\nu}[R^d] = \infty$, we conclude that the series in the right hand side of (25) diverges. By the second Borel-Cantelli lemma (see Durrett [3], page 50), we have that $\mathbf{P}_{p,\nu}(A_k \text{ i.o.}) = 1$. \square

3. PROOF OF THEOREM 7

The proof of Theorem 7 falls naturally into two steps. In the first step, we construct a family of marked point processes $(\mathcal{X}_t, \mathcal{R}_t)$ on \mathbb{Z}^d such that the random graphs $\mathcal{G}(\mathcal{X}_t, \mathcal{R}_t)$ and $\mathcal{G}_{0,t}$ have the same distribution. In the second step, using Theorem 1, we show that for t small enough the connected components of the random graph $\mathcal{G}(\mathcal{X}_t, \mathcal{R}_t)$ are, almost surely, finite. For the sake of clarity, each step is divided into a sequence of lemmas.

In order to prove Theorem 7, we need to introduce some notation. For each $x \in \mathbb{Z}^d$, $r \in \{-1, 0, 1, 2, \dots\}$ and $0 < t$ let

$$N_{x,r}(t) := \sum_{n \geq 1} \mathbf{1}\{T_{x,n} \leq t\} \mathbf{1}\{K_{x,n} > r\}.$$

$N_{x,r}(t)$ is nothing but the number of occurrences of the marked Poisson process $(\mathcal{T}_x, \mathcal{K}_x)$ during the time interval $(0, t]$ whose marks are greater than r . Notice that $N_x(t) := N_{x,-1}(t)$ is the counting measure associated to the Poisson process \mathcal{T}_x .

Let $(\mathcal{T}, \mathcal{K}) = \{(\mathcal{T}_x, \mathcal{K}_x) : x \in \mathbb{Z}^d\}$ be a family of mutually independent marked Poisson point processes on the time line $[0, \infty)$. Let $\mathbf{P}_{(\mathcal{T}, \mathcal{K})}$ and $\mathbf{E}_{(\mathcal{T}, \mathcal{K})}$ respectively be the probability measure and the expectation operator induced by $(\mathcal{T}, \mathcal{K})$.

Remark 12. For each $x \in \mathbb{Z}^d$, let $M_x(t) := \sum_{n \geq 1} \max(K_{x,1}, \dots, K_{x,n}) \mathbf{1}\{N_x(t) = n\}$. It follows from the construction described above and the Coloring Theorem (see Kingman [10], page 52) that

$$(26) \qquad \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(M_x(t) \leq r) = \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(N_{x,r}(t) = 0) = \exp(-M_x t G_x(r)),$$

where $G_x(r) = \sum_{k > r} \nu_x(k)$. Also, we have

$$(27) \qquad \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(M_x(t) \leq r | N_x(t) \geq 1) = \frac{\exp(-M_x t G_x(r)) - \exp(-M_x t)}{1 - \exp(-M_x t)}.$$

3.1. Simultaneous coupling construction: Step 1. Let $\mathcal{U}_1 = (U_{x,1} : x \in \mathbb{Z}^d)$ and $\mathcal{U}_2 = (U_{x,2} : x \in \mathbb{Z}^d)$ be two mutually independent families of independent uniform random variables on $(0, 1]$. The probability space where this coupling is performed is the one where the two families of uniform random variables are defined. We denote by $\mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}$ the probability measure induced by the families of random variables \mathcal{U}_1 and \mathcal{U}_2 .

For each $x \in \mathbb{Z}^d$ and $t > 0$, define

$$(28) \quad X_{x,t} := \mathbf{1}\{U_{x,1} \leq 1 - \exp(-M_x t)\},$$

$$(29) \quad R_{x,t} := F_{x,t}^{-1}(U_{x,2}),$$

where $F_{x,t}(r)$ is the cumulative distribution function given by

$$(30) \quad F_{x,t}(r) = \frac{\exp(-M_x t G_x(r)) - \exp(-M_x t)}{1 - \exp(-M_x t)}.$$

Set $\mathcal{X}_t = (X_{x,t} : x \in \mathbb{Z}^d)$ and $\mathcal{R}_t = (R_{x,t} : x \in \mathbb{Z}^d)$. It follows from the construction that the process $(\mathcal{X}_t, \mathcal{R}_t)$ is a marked point process on \mathbb{Z}^d satisfying

$$(31) \quad \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(X_{x,t} = 1) = \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(N_x(t) \geq 1),$$

$$(32) \quad \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,t} \leq r) = \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(M_x(t) \leq r | N_x(t) \geq 1).$$

Lemma 13. *Let $t > 0$ and let $(\mathcal{X}_t, \mathcal{R}_t)$ be the marked point process on \mathbb{Z}^d defined by (28) and (29). Then, the random graphs $\mathcal{G}(\mathcal{X}_t, \mathcal{R}_t)$ and $\mathcal{G}_{0,t}$ are equally distributed.*

Proof of Lemma 13. By (31) and (32) we have that

$$(33) \quad \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(X_{x,t} = 1, R_{x,t} \leq r) = \mathbf{P}_{(\mathcal{T}, \mathcal{K})}(N_x(t) \geq 1, M_x(t) \leq r).$$

Then, the random graphs $\mathcal{G}_{0,t}(\mathcal{T}, \mathcal{K})$ and $\mathcal{G}(\mathcal{X}_t, \mathcal{R}_t)$ have the same distribution. \square

3.2. Properties of the coupled processes $(\mathcal{X}_t, \mathcal{R}_t)$. Now, we study the main properties of the marked point processes $(\mathcal{X}_t, \mathcal{R}_t)$ needed for the proof of Theorem 7. We begin by proving the following auxiliary result.

Lemma 14. *For any $a \in (0, 1)$, the function*

$$(34) \quad h(z) = \frac{1 - \exp(-az)}{1 - \exp(-z)}$$

is non-decreasing on $[0, \infty)$.

Proof of Lemma 14. It suffices to prove the result for rational a . Then, assume that a is a positive rational number and write it as the ratio of two positive integers $a = m/n$, where $1 < m < n$. Now, making $y = \exp(-z/n)$ we get

$$(35) \quad \frac{1 - \exp(-az)}{1 - \exp(-z)} = \frac{1 - y^m}{1 - y^n} = \left(1 + \frac{\sum_{i=0}^{n-m-1} y^k}{\sum_{k=1}^m y^{-k}}\right)^{-1}.$$

Since the expression above is a decreasing function of y and $y = \exp(-z/n)$ is itself a decreasing function of z , the result follows. \square

Lemma 15. *If $0 < t' < t \leq 1$, then*

$$(36) \quad (\mathcal{X}_{t'}, \mathcal{R}_{t'}) \preceq (\mathcal{X}_t, \mathcal{R}_t).$$

Proof of Lemma 15. Fix $x \in \mathbb{Z}^d$. It follows from (28) that $X_{x,t'} \leq X_{x,t}$. Fix $r \in \mathbb{N}_0$. We shall show that if $0 < t' < t$, then

$$(37) \quad F_{x,t}(r) \leq F_{x,t'}(r).$$

By (30), it suffices to show that

$$(38) \quad \tilde{h}(t) = \frac{1 - \exp(-M_x t G_x(r))}{1 - \exp(-M_x t)}$$

is a non-decreasing function on $(0, 1]$. This follows from Lemma 14 by substituting a by $G_x(r)$ and z by $M_x t$.

From (37) we may conclude that the random variables $R_{x,t}$ defined in (29) satisfy $R_{x,t'} \leq R_{x,t}$. \square

3.3. Simultaneous coupling construction: Step 2. For each $x \in \mathbb{Z}^d$ and $t > 0$, let

$$(39) \quad X_{x,t}^* := \mathbf{1}\{U_{x,1} \leq 1 - \exp(-M^* t)\},$$

$$(40) \quad R_x^* := \sum_{r \in \mathbb{N}_0} r \mathbf{1} \left\{ \inf_{x \in \mathbb{Z}^d} F_{x,1}(r-1) < U_{x,2} \leq \inf_{x \in \mathbb{Z}^d} F_{x,1}(r) \right\}$$

and define $\mathcal{X}_t^* = (X_{x,t}^* : x \in \mathbb{Z}^d)$ and $\mathcal{R}^* = (R_x^* : x \in \mathbb{Z}^d)$.

First, we use conditions (9) and (10) to show that the random variables introduced in (40) are well defined. For that purpose it suffices to show that $\lim_{r \rightarrow \infty} \inf_{x \in \mathbb{Z}^d} F_{x,1}(r) = 1$. By (9), we have

$$\begin{aligned} F_{x,1}(r) &= \frac{\exp(-M_x G_x(r)) - \exp(-M_x)}{1 - \exp(-M_x)} \\ &= 1 - \frac{1 - \exp(-M_x G_x(r))}{1 - \exp(-M_x)} \\ &\geq 1 - \frac{1 - \exp(-M^* \sup_{x \in \mathbb{Z}^d} G_x(r))}{1 - \exp(-M_*)} \end{aligned}$$

for any $x \in \mathbb{Z}^d$. Then,

$$(41) \quad \inf_{x \in \mathbb{Z}^d} F_{x,1}(r) \geq 1 - \frac{1 - \exp(-M^* \sup_{x \in \mathbb{Z}^d} G_x(r))}{1 - \exp(-M_*)}.$$

On the other hand,

$$(42) \quad \sup_{x \in \mathbb{Z}^d} G_x(r) = \sup_{x \in \mathbb{Z}^d} \sum_{\ell > r} \nu_x(\ell) \leq \sum_{\ell > r} \sup_{x \in \mathbb{Z}^d} \nu_x(\ell).$$

Under condition (10) we may conclude that the right hand side of (42) converges to 0 when r goes to ∞ . Then

$$(43) \quad \lim_{r \rightarrow \infty} \sup_{x \in \mathbb{Z}^d} G_x(r) = 0.$$

From (41) and (43) we deduce that $\lim_{r \rightarrow \infty} \inf_{x \in \mathbb{Z}^d} F_{x,1}(r) = 1$.

3.4. The dominating marked point process $(\mathcal{X}_t^*, \mathcal{R}^*)$.

Lemma 16. *For any $0 < t \leq 1$,*

$$(44) \quad (\mathcal{X}_t, \mathcal{R}_t) \preceq (\mathcal{X}_t^*, \mathcal{R}^*).$$

Proof of Lemma 16. It follows from the construction that $X_{x,t} \leq X_{x,t}^*$ and $R_{x,1} \leq R_x^*$. By Lemma 15, we have that $R_{x,t} \leq R_{x,1}$. This completes the proof. \square

The last ingredient needed to prove Theorem 7 is the following result.

Lemma 17. *Under the same assumptions of Theorem 7 there exists $0 < t_0 \leq 1$ such that $\mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(\text{Percolation}) = 0$ for all $0 < t \leq t_0$.*

Proof of Lemma 17. First note that, for each $t > 0$, $(\mathcal{X}_t^*, \mathcal{R}^*)$ is a spatially homogeneous, marked point processes on \mathbb{Z}^d with retention parameter $p(t) = 1 - \exp(-M^*t)$ and probability function of its marks $\nu(r) = \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(\hat{R}_x = r)$ satisfying

$$(45) \quad \sum_{r \in \mathbb{N}_0} r^d \nu(r) < \infty.$$

Indeed,

$$(46) \quad \begin{aligned} \nu(r) &= \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} \leq r) - \inf_{x \in \mathbb{Z}^d} \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} \leq r-1) \\ &\leq \sup_{x \in \mathbb{Z}^d} \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} = r) \end{aligned}$$

Inequality (46) follows from the inequality $\inf\{a_x + b_x\} \leq \inf\{a_x\} + \sup\{b_x\}$ applied to the sequence $a_x(r) = \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} \leq r-1)$ and $b_x(r) = \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} = r)$.

On the other hand, we have

$$(47) \quad \begin{aligned} \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} = r) &= \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} \leq r) - \mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(R_{x,1} \leq r-1) \\ &= \frac{\exp(-M_x G_x(r)) - \exp(-M_x G_x(r-1))}{1 - \exp(-M_x)} \\ &= \frac{\exp(-M_x G_x(r))}{1 - \exp(-M_x)} (1 - \exp(-M_x \nu_x(r))) \\ &\leq \left(\frac{M^*}{1 - \exp(-M_*)} \right) \nu_x(r). \end{aligned}$$

The last inequality follows from well known properties of the exponential function.

From (46) and (47) we may conclude that

$$(48) \quad \nu(r) \leq \left(\frac{M^*}{1 - \exp(-M_*)} \right) \sup_{x \in \mathbb{Z}^d} \nu_x(r).$$

Inequality (45) follows immediately from (48).

Since $(\mathcal{X}_t^*, \mathcal{R}^*)$ satisfies the hypothesis of Theorem 1, there exists $p_0 > 0$ such that $\mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(\text{Percolation}) = 0$ for any $0 < t \leq 1$ such that $p(t) \leq p_0$. Therefore, the connected

components of the random graphs $\mathcal{G}(\mathcal{X}_t^*, \mathcal{R}^*)$ are almost surely finite for any $0 < t \leq \min(-\frac{1}{M^*} \log(1 - p_0), 1)$. Indeed,

$$1 - \exp(-M^* t) \leq p_0 \iff t \leq -\frac{1}{M^*} \log(1 - p_0).$$

□

3.5. Proof of Theorem 7. By Lemma 13, the random graphs $\mathcal{G}_{0,t}(\mathcal{T}, \mathcal{K})$ and $\mathcal{G}(\mathcal{X}_t, \mathcal{R}_t)$ have the same distribution. By Lemmas 16 and 17, we have that there exists $0 < t_0 \leq 1$ such that $\mathbf{P}_{(\mathcal{U}_1, \mathcal{U}_2)}(\text{Percolation}) = 0$ for all $0 < t \leq t_0$. Therefore, the connected components of the Harris random graph $\mathcal{G}_{0,t}(\mathcal{T}, \mathcal{K})$ are almost surely finite for all $0 < t \leq t_0$. □

4. APPENDIX

4.1. Harris graphical construction. Let $t_0 > 0$ be as in Theorem 7 and let $C_\ell = C_\ell(\mathcal{T} \cap [0, t_0], \mathcal{K})$, $\ell \in \mathbb{N}$, be the partition of \mathbb{Z}^d into a countable number of finite islands, with no interaction between them.

Finite-volume construction. The construction of the process on each finite island C_ℓ with initial configuration η using the Poisson processes is straightforward because the epoch of the associated Poisson processes are well ordered.

Fix $\ell \in \mathbb{N}$ and consider $0 < \tau_1 < \tau_2 < \dots < \tau_n$, where

$$(49) \quad \{\tau_1, \tau_2, \dots, \tau_n\} = \bigcup_{x \in C_\ell} (\mathcal{T}_x \cap [0, t_0]).$$

We construct the process $\sigma_t^\eta(C_\ell)$ inductively as follows. For $k = 1, \dots, n$, let x_1, x_2, \dots, x_n be the sites such that $\tau_k \in \mathcal{T}_{x_k}$.

Step 1. Let

$$\sigma_t^\eta(C_\ell) = \eta(C_\ell) \text{ for all } 0 \leq t < \tau_1$$

and set

$$\sigma_{\tau_1}^\eta(C_\ell) = \sigma_{x_1, W_{x_1}(\sigma_{\tau_1-}^\eta(C_\ell))}^\eta.$$

Thus we have defined the process on the time interval $[0, \tau_1]$.

Inductive step. Assume that $\sigma_t^\eta(C_\ell)$ has already been defined for all $0 \leq t \leq \tau_k$. Then set

$$\sigma_t^\eta(C_\ell) = \sigma_{\tau_k}^\eta(C_\ell) \text{ for all } \tau_k < t < \tau_{k+1}$$

and

$$\sigma_{\tau_{k+1}}^\eta(C_\ell) = \sigma_{x_{k+1}, W_{x_{k+1}}(\sigma_{\tau_{k+1}-}^\eta(C_\ell))}^\eta.$$

This step is repeated until the construction has been finished on the time interval $[0, t_0]$. Infinite-volume construction. Finally, for each $\ell \in \mathbb{N}$, let

$$(50) \quad (\sigma_t^\eta)(C_\ell) := \sigma_t^\eta(C_\ell), \quad t \in (0, t_0].$$

Since t_0 is independent of the initial configuration, we may conclude, by means of the Markovian property of Poisson processes, that the state of the process may be computed by induction at any time $t \geq 0$.

4.2. Geometry of \mathbb{Z}^d . The following proposition deals with geometric aspects used in the proof of Lemma 9.

Proposition 18. *Fix $d \in \mathbb{N}$. Then, for any $n, r \in \mathbb{N}$, we have*

$$S_{nr} \subset \bigcup_{x \in S_n} B\left(rx, \frac{d}{2}r\right).$$

In order to prove Proposition 18 we need the following result.

Lemma 19. *Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ be such that $1 > x_1 \geq x_2 \geq \dots \geq x_d > 0$ and $\sum_{i=1}^d x_i = m \in \mathbb{N}$, where $m < d$. Let $y = (1, \dots, 1, 0, \dots, 0)$ (m -ones). Then, $\|y - x\| \leq \frac{d}{2}$.*

Proof of Proposition 18. By symmetry, it suffices to prove the proposition for x in the region $\{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, 2, \dots, d\}$. Fix n and $r \in \mathbb{N}$. By Lemma 19, we have that for any $x \in \mathbb{R}^d$ with $\|x\| = n$ there exists $y \in \mathbb{Z}^d$ with $\|y\| = n$ such that $\|y - x\| \leq \frac{d}{2}$.

Pick $x \in S_{nr}$. Then, for $\frac{x}{r} \in S_n$ there exists $y \in \mathbb{Z}^d$ with $\|y\| = n$ such that $\|y - \frac{x}{r}\| \leq \frac{d}{2}$. Thus, $\|ry - x\| \leq \frac{d}{2}r$. \square

We finish this subsection by proving Lemma 19.

Proof. We begin by observing that

$$\begin{aligned} \|y - x\| &= \sum_{i=1}^m (1 - x_i) + \sum_{i=m+1}^d x_i \\ &= 1 - x_1 + \sum_{i=2}^m (1 - x_i) + \sum_{i=m+1}^d x_i \\ &= 1 - m + \sum_{i=2}^d x_i + m - 1 - \sum_{i=2}^m x_i + \sum_{i=m+1}^d x_i \\ (51) \quad &= 2 \sum_{i=m+1}^d x_i. \end{aligned}$$

Now assume that $\|y - x\| > \frac{d}{2}$. From (51) we have that $\sum_{i=m+1}^d x_i > \frac{d}{4}$. Therefore, there exists $i \in \{m+1, \dots, d\}$ such that $x_i > \frac{d}{4(d-m)}$. Also, $x_1, \dots, x_m > \frac{d}{4(d-m)}$. Since $\sum_{i=1}^m x_i > \frac{dm}{4(d-m)}$ we get

$$(52) \quad \sum_{i=1}^d x_i > \frac{dm}{4(d-m)} + \frac{d}{4} = \frac{d^2}{4(d-m)}.$$

Note that

$$(53) \quad \frac{d^2}{4(d-m)} \geq m$$

if, and only if, $(d - 2m)^2 \geq 0$. Since the last inequality is true, so is inequality (53). From (52) and (53) we get that $\sum_{i=1}^d x_i > m$ which is a contradiction. The contradiction comes from the fact that we have assumed that $\|y - x\| > \frac{d}{2}$. \square

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REFERENCES

- [1] Bramson, M.; Kalikow, S. (1993) Non uniqueness in g-functions. *Isr. J. Math.* 84, No. 1-2, 153-160
- [2] Durrett, R. (1995) Ten lectures on particle systems. *Lecture Notes in Mathematics* 1608 97-207, Springer Verlag, Berlin-Heidelberg-New-York.
- [3] Durrett, R. (1996) *Probability: theory and examples* (Second Edition). Duxbury Press, Belmont.
- [4] Galves, A; Garcia, N. L.; Lcherbach, E.. Perfect simulation and finitary coding for multicolor systems with interactions of infinite range. *arXiv*, (2008).
- [5] Gouere, J. B (2008). Subcritical regimes in the Poisson Boolean model of continuum percolation. *Ann. Probab.* 36, No. 4 1209-1220. MR2435847
- [6] Grimmett, G. (1999). *Percolation* (Second Edition). Springer, Berlin.
- [7] Hall, P. (1985). On continuum percolation. *Ann. Probab.* 13 1250-1266. MR0806222
- [8] Harris, T. E. (1972) Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.* 9 66-89.
- [9] Kalikow, S. (1990) Random Markov processes and uniform martingales. *Isr. J. Math.* 71, No.1, 33-54
- [10] Kingman, J.F.C., *Poisson Processes*. Oxford University Press, New York (2002).
- [11] Liggett T.M., *Interacting Particle Systems*. Springer, Berlin (1985).
- [12] Meester, R. and Roy, R. (1996). *Continuum Percolation*. Cambridge Univ. Press. MR1409145
- [13] Seppalainen, T. (2008). *Translation Invariant Exclusion Process*. Book in Progress. Available at www.math.wisc.edu/~seppalai/excl-book/ajo.pdf

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